Inversion of a "discontinuous coordinate transformation" in general relativity

Evelina Erlacher*, Michael Grosser*

Abstract

In [21], Penrose—in a purely formal way—introduced a "discontinuous coordinate transformation", which relates a continuous representation of the metric of impulsive pp-waves to a discontinuous one. On the basis of the invertibility concept for generalized functions developed recently by the first author in [10], we show that this discontinuous coordinate transformation indeed represents an invertible generalized function in the appropriate sense.

Keywords: discontinuous coordinate transformation, impulsive pp-wave, distributional metric, Colombeau algebra, inverse generalized function

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1 Introduction

In general relativity, so-called impulsive pp-waves have been described by two different metrics: In [21, Chapter 4], Penrose used the form

$$\mathrm{d}s^2 = f(x,y)\,\delta(u)\,\mathrm{d}u^2 - \mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{1}$$

where δ denotes the Dirac delta distribution. This space-time is flat everywhere except for the null hyperplane $u = 0$ where the curvature is concentrated. On the other hand, impulsive pp-waves have also been described by the continuous metric

$$\begin{aligned}\mathrm{d}s^2 = -\mathrm{d}u\,\mathrm{d}V &+ (1+\frac{1}{2}\,\partial_{11}f\,u_+)^2\,\mathrm{d}X^2 + (1+\frac{1}{2}\,\partial_{22}f\,u_+)^2\,\mathrm{d}Y^2 \\ &+ \frac{1}{2}\,\partial_{12}f\,\Delta f\,u_+^2\,\mathrm{d}X\,\mathrm{d}Y + 2u_+\,\partial_{12}f\,\mathrm{d}X\,\mathrm{d}Y + \frac{1}{4}(\partial_{12}f)^2\,u_+^2\,(\mathrm{d}X^2+\mathrm{d}Y^2)\end{aligned} \tag{2}$$

(see formula (17) in [1]) where for simplicity we have suppressed the dependence of the function f on its arguments, i.e., f is to be read as $f(X,Y)$. u_+ denotes the kink function on $\mathbb{R}$, vanishing on the negative axis and acting as identity on the positive axis.

*Fakultät für Mathematik, Universität Wien, Nordbergstr. 15, A-1090 Wien, Austria, e-mail: evelina.erlacher@univie.ac.at, michael.grosser@univie.ac.at

Since both (1) and (2) aim at modelling the same physical (though idealized) situation, they have to be viewed as equivalent from a physical point of view. Therefore it seems plausible that a coordinate transformation can be found relating (u, x, y, v) and (u, X, Y, V) such that the respective substitutions transform (1) and (2) into each other. Of course, since the coefficients in (2) are continuous while (1) contains the delta distribution, such a transformation cannot even be continuous; strictly speaking, it has to change the topological structure of the manifold.

A transformation connecting (1) and (2) indeed exists: Formally, it arises from the "distributional geodesics" of the metric (1), with vanishing initial speed in the x, y and v-directions. It has been given by Penrose in [21] for the special case $f(x, y) = x^2 - y^2$. For general f, it appears as formula (16) in [1] and has the form

$$
\begin{aligned}
u &= u, \\
x^i &= X^i + \frac{1}{2}\,\partial_i f(X^k)\,u_+, \\
v &= V + f(X^k)\,H(u) + \frac{1}{4}\sum_{i=1}^{2}\partial_i f(X^k)^2\,u_+
\end{aligned}
\tag{3}
$$

where we write (X^k) for $(X^1, X^2) = (X, Y)$ and $(x^1, x^2) = (x, y)$. Due to the occurrence of the step function (Heaviside function) H, this "coordinate transformation" obviously is discontinuous. Formulas (2) and (3) have already been alluded to implicitly in [21, Chapter 4], yet with only the special case $f(x, y) := x^2 - y^2$ actually written down.

From a mathematical point of view, it certainly seems desirable to embed this discontinuous transformation into a suitable theoretical frame of generalized functions. Many important concepts which nowadays belong to the core of rigorous standard mathematics had their origin and their "prehistory" in ingenious formal calculations of physicists. Distributions as we know them today might serve as a paradigmatic example. To give a precise meaning to the intuitive idea of a discontinuous coordinate transform, a conceptual setting of generalized functions is required which admits composition, hence nonlinear operations, and defining inverses. These requirements immediately rule out linear distribution theory.

The nonlinear theory of generalized functions going back to J.F. Colombeau, however, at least provides sufficiently broad concepts of composition (cf.[13, 1.2.8, 1.2.29]). However, there remain severe difficulties as to developing a useful notion of inversion which mainly are due to the lack of a sensible notion of range or image of a set under a generalized function. This image set, of course, would have to serve as the domain for any presumptive inverse (see [10, Section 1] for a more extensive discussion). Even for the simple case of a step function (occurring in (3), after all), the mathematical status of an "inverse" is by no means clear.

Nevertheless, Kunzinger and Steinbauer tackled this problem in [15] and [26]; see also [13, Section 5.4.3]. They succeeded in giving a more precise meaning to the intuitive resp. purely formal idea of equivalence of the two descriptions of impulsive pp-waves by interpreting the discontinuous transformation as the distributional shadow of a generalized transformation: After regularizing the

distributional space-time metric (1), they applied a generalized change of coordinates modelling the distributional one; then they calculated the distributional shadow of the transformed generalized metric to arrive precisely at the continuous form (2) (cf. [15, 13]). However, their claim that the "generalized Penrose transformation" envisaged above indeed represents a Colombeau generalized function having the required properties as to domain and c-boundedness (see Definition 3.2) cannot be maintained as stated in [13, Theorem 5.7.3 and its proof]. Furthermore, the question to what extent the generalized function representing the discontinuous coordinate transformation is "invertible" could not be answered (in fact, not even be posed in a precise sense) at that time due to the lack of an appropriate notion of an inverse of a (Colombeau) generalized function. Thus, Kunzinger and Steinbauer's analysis of the generalized coordinate transformation as an invertible Colombeau function had to remain incomplete. At the appropriate places in the subsequent sections we will review their achievements in more detail.

Recently, the first author has presented a conceptual frame for viewing certain Colombeau generalized functions as invertible, together with necessary resp. sufficient conditions for invertibility (cf. [10]). It is the main purpose of this paper to show that the discontinuous coordinate transformation connecting (1) and (2) indeed represents an invertible generalized function in the appropriate sense.

This paper is organized as follows: Section 2 collects some background information on pp-waves while Section 3 provides the basic terminology for Colombeau algebras, covering, in particular, c-boundedness and invertibility of Colombeau functions. The principal steps towards the main result of this article are reflected by the pattern of Sections 4–8: The construction of Kunzinger and Steinbauer ([16, 15] resp. [13]) yielding a generalized solution of the regularized geodesic equation corresponding to the metric (1) is reviewed and complemented in Section 4. From these geodesics, we obtain the generalized coordinate transformation $T := [(t_\varepsilon)_\varepsilon]$ which will be established as a c-bounded Colombeau generalized function from $\mathbb{R}^4$ to $\mathbb{R}^4$ in Section 5. Section 6 again builds on and extends results of Kunzinger and Steinbauer [15, 13] concerning injectivity of t_ε and the "strict non-zeroness" of the Jacobian determinant of t_ε. Both these properties (to be satisfied on sufficiently large sets) will be crucial for showing the invertibility of T in the final section. We provide the necessary information on size and shape of the sets of injectivity and their dependence on the relevant parameters. Now the main difficulty in establishing the (local) invertibility of T consists in proving that the images of sufficiently large open sets (fixed with respect to ε) under the maps t_ε intersect with non-empty interior, for ε small. This intersection will then serve to obtain the domains for local inverses of the t_ε and, in the sequel, also for the inverse generalized function. By means of a non-trivial result from [10] on the stability of image sets under injective continuous functions, this intersection is shown in Section 8 to be non-empty indeed. Putting together all the pieces, we finally obtain local invertibility of T in Theorem 8.5. The results of Section 7 prepare the ground for the final section by providing some convergence relations needed for applying the stability theorem.

2 Plane fronted gravitational waves with parallel rays (pp-waves)

This section collects some basic facts on pp-waves, together with corresponding references.

The line elment of a plane fronted gravitational wave with parallel rays (a space-time characterized by the existence of a covariantly constant null vector field) or, for short, a pp-wave can be written in the form

$$\mathrm{d}s^2 = h(u,x,y)\,\mathrm{d}u^2 - \mathrm{d}u\,\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2, \tag{4}$$

where h—the wave profile—is an arbitrary smooth function of the retarded time coordinate u and the Cartesian coordinates x, y spanning the wave surface. [13, Subsection 5.3.1] surveys various aspects of pp-waves and provides numerous references.

If the wave profile is given by $h(u,x,y) = f(x,y)\,\delta(u)$ for f an arbitrary smooth function and δ the Dirac-δ (cf. (1)) the corresponding space-times are called *impulsive* pp-waves. Penrose introduced such space-times as limits of suitable sequences of sandwich waves (cf. [20]). Moreover, they naturally arise in a number of situations, e.g. as ultrarelativistic limits of boosted black hole geometries of the Kerr-Newman familiy [2, 4, 17], as multipole solutions of the Weyl family [22], and in particle scattering at the Planck scale [29, 7].

Various aspects of impulsive pp-waves have been discussed by several authors. Let us mention [1] and [23] for continuous forms of the metric, the "scissors and paste approach" of Penrose in [21] and the work [8] of Dray and t'Hooft.

The obvious disadvantage of a description of impulsive pp-waves by (1) is the occurrence of distributional coefficients in the metric and, consequently, also in the corresponding geodesic equations given by

$$\begin{aligned} \ddot{x}^i(u) &= \frac{1}{2}\,\partial_i f(x^1(u), x^2(u))\,\delta(u), \\ \ddot{v}(u) &= f(x^1(u), x^2(u))\,\dot{\delta}(u) + 2\sum_{i=1}^{2} \partial_i f(x^1(u), x^2(u))\,\dot{x}^i(u)\,\delta(u), \end{aligned} \tag{5}$$

(cf. [25] for their derivation). The right hand side of the equation for v involves the product of δ and the Heaviside function H (due to $x^i(u)$ involving the kink function u_+, cf. [13, Theorem 5.3.3]) which is not defined in the linear theory of distributions. Nevertheless, attempts have been made to solve the system (5) (though ill-defined) in $\mathcal{D}'$ (cf. [11, 3]) by simply setting $H\delta = \frac{1}{2}\delta$. Such *ad hoc* multiplication rules may work out in certain instances (such as this one, cf. [25, 16, 13]) but in just as many cases they will lead to considerable difficulties (cf. e.g. [14, 24]). The nonlinear theory of generalized functions as introduced by J. F. Colombeau (see [5, 6, 19, 13]) provides a setting where these problems can be overcome in a rigorous mathematical fashion without the need for imposing such multiplication rules. Indeed, Kunzinger and Steinbauer presented a method of treating equations such as (5) in a mathematically satisfactory way (see [25, 16, 15, 13]): They regularized the given equations, solved them in a suitable Colombeau algebra and showed that the solutions indeed possess regularization-independent distributional limits coinciding with the distributional "solutions" given in [11] and [3].

There have been a number of further successful applications of Colombeau theory to general relativity. For an extensive list of references, we refer to the one at the end of the survey article [27] of Steinbauer and Vickers. The recent paper [28] compares two different approaches to metrics of low differentiability in general relativity.

3 Notation and preliminaries

For subsets A, B of a topological space X, we write $A \subset\subset B$ if A is a compact subset of the interior B° of B. For each non-empty open subset of U of $\mathbb{R}^n$ we denote by $\mathcal{D}(U)$ the linear space of test functions on U, i.e., of infinitely differentiable real-valued functions having compact support in U.

Concerning fundamentals of (special) Colombeau algebras, we follow [13, Subsection 1.2]. As to inversion of generalized functions, we adopt terminology and results from [10].

In particular, for defining the special Colombeau algebra $\mathcal{G}(U)$ on a given (non-empty) open subset U of $\mathbb{R}^n$, we set $\mathcal{E}(U) := \mathrm{C}^\infty(U, \mathbb{R})^{(0,1]}$ and

$$\begin{aligned}
\mathcal{E}_M(U) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(U) \,|\, \forall\, K \subset\subset U \;\forall\, \alpha \in \mathbb{N}_0^n \;\exists\, N \in \mathbb{N} : \\
\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}, \\
\mathcal{N}(U) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}(U) \,|\, \forall\, K \subset\subset U \;\forall\, \alpha \in \mathbb{N}_0^n \;\forall\, m \in \mathbb{N} : \\
\sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}.
\end{aligned}$$

Elements of $\mathcal{E}_M(U)$ resp. $\mathcal{N}(U)$ are called *moderate* resp. *negligible functions*. $\mathcal{E}_M(U)$ is a subalgebra of $\mathcal{E}(U)$, $\mathcal{N}(U)$ is an ideal in $\mathcal{E}_M(U)$. The *special Colombeau algebra* on U is defined as

$$\mathcal{G}(U) := \mathcal{E}_M(U)/\mathcal{N}(U).$$

The class of a moderate net $(u_\varepsilon)_\varepsilon$ in this quotient space will be denoted by $[(u_\varepsilon)_\varepsilon]$. A generalized function on some open subset U of $\mathbb{R}^n$ with values in $\mathbb{R}^m$ is given as an m-tuple $(u_1, \cdots, u_m) \in \mathcal{G}(U)^m$ of generalized functions $u_j \in \mathcal{G}(U)$ where $j = 1, \cdots, m$.

The composition $v \circ u$ of two arbitrary generalized functions is not defined, not even if v is defined on the whole of $\mathbb{R}^m$ (i.e., if $u \in \mathcal{G}(U)^m$ and $v \in \mathcal{G}(\mathbb{R}^m)^p$). A convenient condition for $v \circ u$ to be defined is to require u to be "compactly bounded" (c-bounded) into the domain of v. Since there is a certain inconsistency in [13] concerning the precise description of c-boundedness (see [10, Section 2] for details) we include the explicit definition of this important property below. For a full discussion, see again [10, Section 2].

3.1. Definition. *Let U and V be open subsets of $\mathbb{R}^n$ resp. $\mathbb{R}^m$. An element $(u_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(U, V)^{(0,1]}$ is called compactly bounded (c-bounded) if the conditions*

(1) For every $K \subset\subset U$ there exist $L \subset\subset V$ and $\varepsilon_0 \in (0, 1]$ such that $u_\varepsilon(K) \subseteq L$ for all $\varepsilon \leq \varepsilon_0$.

(2) *For every $K \subset\subset U$ and every $\alpha \in \mathbb{N}_0^m$ there exists $N \in \mathbb{N}$ with*

$$\sup_{x \in K} |\partial^\alpha u_\varepsilon^j(x)| = O(\varepsilon^{-N})$$

for all component functions u_ε^j $(j = 1, \ldots, m)$ of u_ε.

are satisfied. The collection of c-bounded elements of $\mathrm{C}^\infty(U, V)^{(0,1]}$ is denoted by $\mathcal{E}_M[U, V]$.

Obviously, $\mathcal{E}_M[U, V]$ can be viewed as a subset of $\mathcal{E}_M(U)^m$ and thus determines a certain subset of $\mathcal{G}(U)^m$.

3.2. Definition. *Let U and V be open subsets of $\mathbb{R}^n$ resp. $\mathbb{R}^m$.*

(1) *An element $(u_\varepsilon)_\varepsilon$ of $\mathcal{E}_M(U)^m$ is called c-bounded from U into V if, in fact, $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M[U, V]$.*

(2) *An element u of $\mathcal{G}(U)^m$ is called c-bounded from U into V if it has a representative which is c-bounded from U into V, i.e., which is a member of $\mathcal{E}_M[U, V]$. The space of all c-bounded generalized functions from U into V will be denoted by $\mathcal{G}[U, V]$.*

Due to the asymptotic nature of the conditions defining $\mathcal{E}_M(U)$ resp. $\mathcal{N}(U)$, the property of a generalized function $u \in \mathcal{G}(U)^m$ to be c-bounded from U into V is not affected if we only require the existence of a representative $(u_\varepsilon)_\varepsilon$ satisfying $u_\varepsilon(U) \subseteq V$ and conditions 3.2 (1) and 3.2 (2) for all ε below some $\varepsilon_1 > 0$ depending on the net at hand.

3.3. Proposition. *Let $u \in \mathcal{G}(U)^m$ be c-bounded into V and let $v \in \mathcal{G}(V)^p$, with representatives $(u_\varepsilon)_\varepsilon$ resp. $(v_\varepsilon)_\varepsilon$. Then the composition*

$$v \circ u := [(v_\varepsilon \circ u_\varepsilon)_\varepsilon]$$

is a well-defined generalized function in $\mathcal{G}(U)^p$.

Next, we present the notions of invertibility introduced in [10].

3.4. Definition (Invertibility of generalized functions). *Let U be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}(U)^n$. Let G be an open subset of U.*

(LI) *u is called left invertible on G if there exist $v \in \mathcal{G}(V)^n$ with V an open subset of $\mathbb{R}^n$ and an open set $H \subseteq V$ such that $u|_G$ is c-bounded into H and $v \circ u|_G = \mathrm{id}_G$. Then v is called a left inverse of u on G.*

In shorthand, u is left invertible (on G) with left inversion data $[G, V, v, H]$.

(RI) *u is called right invertible on G if there exist $v \in \mathcal{G}(V)^n$ with V an open subset of $\mathbb{R}^n$ and an open set $H \subseteq V$ such that $v|_H$ is c-bounded into G and $u \circ v|_H = \mathrm{id}_H$. Then v is called a right inverse of u on G.*

In shorthand, u is right invertible (on G) with right inversion data $[G, V, v, H]$.

(I) *u is called invertible on G if it is both left and right invertible on G with left inversion data $[G, V, v, H_\mathrm{l}]$ and right inversion data $[G, V, v, H_\mathrm{r}]$. Then v is called an inverse of u on G.*

In shorthand, u is invertible (on G) with inversion data $[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$.

(SI) *u is called strictly invertible on G if it is invertible on G with inversion data $[G, V, v, H, H]$ for some open subset H of V. Then v is called a strict inverse of u on G.*

In shorthand, u is strictly invertible (on G) with inversion data $[G, V, v, H]$.

Throughout this paper we will also use the phrases "u is invertible (on G) by $[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$" resp. "$[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$ is an inverse of u (on G)". If we do not specify a set on which a given $u \in \mathcal{G}(U)^n$ is invertible, we always refer to invertibility on U, i.e. on the whole of its domain. The same applies to the cases of "left invertible", "right invertible" resp. "strictly invertible".

Some basic properties of the invertibility concepts introduced above are discussed in [10, Section 3].

3.5. Definition. *Let U be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}(U)^n$. We call u locally (left, right) invertible if for every point $z \in U$ there exists an open neighbourhood G of z in U such that u is (left, right) invertible on G.*

3.6. Definition. *Let U be an open subset of $\mathbb{R}^n$. A moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)$ is called strictly non-zero if for every compact subset K of U there exist $C > 0$, a natural number N and some $\varepsilon_0 \in (0, 1]$ such that*

$$\inf_{x \in K} |u_\varepsilon(x)| \geq C\varepsilon^N \tag{6}$$

for all $\varepsilon \leq \varepsilon_0$. An element u of $\mathcal{G}(U)$ is called strictly non-zero if it possesses a representative with this property.

By [13, Theorem 1.2.5], $u \in \mathcal{G}(U)$ is strictly non-zero if and only if there exists $v \in \mathcal{G}(U)$ with $uv = 1$.

4 Description of the geodesics for impulsivse pp-waves as Colombeau generalized functions

According to [15, p. 1256] resp. [13, p. 464], the transformation connecting the distributional and the continuous forms of the metric ((1) resp. (2)) arises from certain geodesics with respect to (1). This holds true for the regularized resp. generalized versions t_ε as well as—though only on a formal level—for the distributional version t given by (3). In this section, therefore, we study the geodesic equations corresponding to the regularization of the distributional metric (1), following the approach taken in [16, 15] resp. [13]. We include the results obtained by Kunzinger and Steinbauer, establishing existence and uniqueness of the generalized geodesics. In view of the ultimate goal of this article, however, a more refined study of these geodesics is required.

Following [15], we introduce the notion of a strict delta net as follows:

4.1. Definition. *A strict delta net is a net $(\delta_\varepsilon)_\varepsilon$ in $\mathcal{D}(\mathbb{R}^n)$ satisfying*

(1) $\operatorname{supp}(\delta_\varepsilon) \subseteq [-\varepsilon, \varepsilon]$,

(2) $\int \delta_\varepsilon(x)\, dx \to 1$ *for* $\varepsilon \to 0$,

(3) $\int |\delta_\varepsilon(x)|\, dx \leq C$ *for some* $C > 0$ *and small* ε.

A strict delta function is a generalized function $D = [(\delta_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R}^n)$ with $(\delta_\varepsilon)_\varepsilon$ a strict delta net.

Corresponding to (1), we define the generalized metric $\hat{g}$ on $\mathbb{R}^4$ by

$$\hat{\mathrm{d}}s^2 = f(x^1, x^2)\, D(u)\, \mathrm{d}u^2 - \mathrm{d}u\, \mathrm{d}v + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2, \tag{7}$$

where D is a strict delta function. Then, in terms of generalized functions, the geodesic equations (5) take the following form:

$$\begin{aligned} \ddot{x}^i(u) &= \frac{1}{2}\, \partial_i f(x^1(u), x^2(u))\, D(u), \\ \ddot{v}(u) &= f(x^1(u), x^2(u))\, \dot{D}(u) + 2 \sum_{i=1}^{2} \partial_i f(x^1(u), x^2(u))\, \dot{x}^i(u)\, D(u). \end{aligned} \tag{8}$$

At the level of representatives and for fixed ε, the solution of this system is obtained by means of [13, Lemma 5.3.1] (resp. [25, Appendix]). For the convenience of the reader and to facilitate the analysis of the dependence of the domains of the solutions on the initial values, we state this lemma below. The initial conditions are chosen at $u = -1$, i.e. "long before the shock".

4.2. Lemma. *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R} \to \mathbb{R}^n$ be smooth and $(\delta_\varepsilon)_\varepsilon$ a net of smooth functions satisfying conditions 4.1 (1) and 4.1 (3) as above. For any $x_0, \dot{x}_0 \in \mathbb{R}^n$ and any $\varepsilon \in (0, 1]$ consider the system*

$$\begin{aligned} \ddot{x}_\varepsilon(t) &= g(x_\varepsilon(t))\delta_\varepsilon(t) + h(t) \\ x_\varepsilon(-1) &= x_0 \\ \dot{x}_\varepsilon(-1) &= \dot{x}_0. \end{aligned} \tag{9}$$

Let $b > 0$, $Q := \int_{-1}^{1} \int_{-1}^{s} |h(r)| dr\, ds$, $I := \{x \in \mathbb{R}^n \mid |x - x_0| \leq b + |\dot{x}_0| + Q\}$ and

$$\alpha := \min\left(\frac{b}{C\|g\|_{\infty,I} + |\dot{x}_0|}\, ,\, \frac{1}{2LC}\, ,\, 1 \right),$$

with L a Lipschitz constant for g on I. Then (9) has a unique smooth solution x_ε on $J_\varepsilon := [-1, \alpha - \varepsilon]$. Furthermore, for ε sufficiently small (e.g. $\varepsilon \leq \frac{\alpha}{2}$) x_ε is globally defined and both $(x_\varepsilon)_\varepsilon$ and $(\dot{x}_\varepsilon)_\varepsilon$ are bounded on compact subsets of $\mathbb{R}$, uniformly in ε for small ε.

The proof of the uniqueness part of [13, Lemma 5.3.1] actually has to be complemented by an additional argument since, in fact, it only establishes uniqueness of x_ε as an element of $X_\varepsilon := \{z \in \mathrm{C}(J_\varepsilon, \mathbb{R}^n) \mid |z(t) - x_0| \leq b + |\dot{x}_0| + Q\}$ (this is duly taken account of in [9, Lemma 4.2]). Assuming y_ε to be any solution of (9), let $[-1, t_1]$ be the maximal subinterval of $[-1, \alpha - \varepsilon]$ on which $|y_\varepsilon(t) - x_0|$ is bounded by $b + |\dot{x}_0| + Q$. By integrating the differential equation twice within $[-\varepsilon, t_1]$, the assumption $t_1 < \alpha - \varepsilon$ leads to a contradiction. Therefore, $y_\varepsilon \in X_\varepsilon$ and, consequently, $y_\varepsilon = x_\varepsilon$.

For fixed initial values x_0, $\dot{x}_0$ and for small ε (say, $\varepsilon \leq \frac{\alpha}{2}$, with α depending on $\dot{x}_0$ and x_0 via I, L and $\|g\|_{\infty,I}$), the preceding lemma ensures the existence of a solution of the geodesic equations (8), defined on $\mathbb{R}$. This was exploited successfully in [25, 16, 15] resp. [13]. However, in view of our ultimate goal

of establishing the generalized coordinate transformation (3) induced by the generalized geodesics as an invertible generalized function in the sense of [10], a closer analysis of the role played by x_0 and $\dot{x}_0$ in Lemma 4.2 is required: Viewing the solutions x_ε of (9) as functions of $(\varepsilon, x_0, \dot{x}_0, t)$, Lemma 4.2 yields domains of the form $\bigcup_{x_0,\dot{x}_0\in\mathbb{R}^n}(0, \frac{\alpha(x_0,\dot{x}_0)}{2}] \times \{x_0, \dot{x}_0\} \times \mathbb{R}$, and bounds for x_ε depending on $x_0, \dot{x}_0$ via several intermediate steps. To establish our main result, however, we need uniformity of the domain of the solutions, as well as uniformity of bounds for x_ε. Here, uniformity is to be understood as uniformity with respect to x_0 and $\dot{x}_0$ ranging over compact subsets of $\mathbb{R}^n$. This necessary upgrading of Lemma 4.2 is accomplished by Propositions 4.3 (uniformity of domains) and 4.4 (uniformity of bounds) below.

The sets I and J_ε as well as the constants α and L depend on the initial values x_0 and $\dot{x}_0$. Nevertheless, they can be chosen uniformly for $(x_0, \dot{x}_0)$ ranging over some compact set $K \subset\subset \mathbb{R}^{2n}$: For $\beta(K) := \sup_{z\in \mathrm{pr}_2(K)} |z|$, set $I(K) := \mathrm{pr}_1(K) + \overline{\mathrm{B}_{b+\beta(K)+Q}(0)}$, $L(K) := \max_{z\in I(K)} \|\mathrm{D}g(z)\|$, $\alpha(K)$ as in Lemma 4.2 (replacing I, $|\dot{x}_0|$, L by $I(K)$, $\beta(K)$, $L(K)$, respectively) and, finally, $J_\varepsilon(K) := [-1, \alpha(K) - \varepsilon]$. Hence, for $\varepsilon \le \varepsilon(K) := \frac{\alpha(K)}{2}$ and $(x_0, \dot{x}_0) \in K$, the solutions $x_\varepsilon(x_0, \dot{x}_0)$ are globally defined. Note that $\beta(K)$, $I(K)$, $L(K)$ are monotonically increasing with K; $\alpha(K)$ and $J_\varepsilon(K)$ are decreasing as K increases. By the Existence and Uniqueness Theorem for ODEs, x_ε also depends smoothly on the initial values, i.e. $x_\varepsilon \in \mathrm{C}^\infty(K^\circ \times \mathbb{R})$ for $K \subset\subset \mathbb{R}^{2n}$ and $\varepsilon \le \varepsilon(K)$.

4.3. Proposition. *There exists* $(x_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)^{(0,1]}$ *such that for every* $K \subset\subset \mathbb{R}^{2n}$ *there exists* $\varepsilon_K \in (0, 1]$ *such that* $x_\varepsilon(x_0, \dot{x}_0, \,.\,)$ *is the global solution of* (9) *for all* $(x_0, \dot{x}_0) \in K$ *and* $\varepsilon \le \varepsilon_K$. *Moreover,* $\varepsilon_K \le \frac{1}{2}\alpha(x_0, \dot{x}_0)$ *for all* $(x_0, \dot{x}_0) \in K$.

Proof. Let $(K_m)_m$ be an increasing sequence of compact subsets of $\mathbb{R}^{2n}$ satisfying $K_m \subset\subset K^\circ_{m+1}$ which exhausts $\mathbb{R}^{2n}$. Set $A_m := (\varepsilon(K_{m+1}), \varepsilon(K_m)] \times K_m$ and $A := \bigcup_{m=1}^\infty A_m$. Now, we may define a function $y : A \to \mathrm{C}^\infty(\mathbb{R}, \mathbb{R}^n)$, $(\varepsilon, x_0, \dot{x}_0) \mapsto y_\varepsilon(x_0, \dot{x}_0, \,.\,)$ such that $y_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is the global solution of (9). Let $\sigma_m \in \mathcal{D}(K^\circ_m)$ such that $0 \le \sigma_m \le 1$ and $\sigma_m|_{K_{m-1}} = 1$. For $\varepsilon \in (\varepsilon(K_{m+1}), \varepsilon(K_m)]$ (note that $\varepsilon(K_m) \searrow 0$ as $m \to \infty$) we define

$$x_\varepsilon(x_0, \dot{x}_0, t) := \begin{cases} \sigma_m(x_0, \dot{x}_0) \cdot y_\varepsilon(x_0, \dot{x}_0, t), & (x_0, \dot{x}_0) \in K^\circ_m \\ 0, & (x_0, \dot{x}_0) \in \mathbb{R}^{2n} \setminus \operatorname{supp} \sigma_m \end{cases} .$$

Then $x_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and $x_\varepsilon|_{K_{m-1}\times\mathbb{R}} = y_\varepsilon|_{K_{m-1}\times\mathbb{R}}$. Since for $\varepsilon \in (0, \varepsilon(K_m)]$ and $(x_0, \dot{x}_0) \in K_m$ the function $y_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is a global solution, $x_\varepsilon(x_0, \dot{x}_0, \,.\,)$ is a global solution for $\varepsilon \in (0, \varepsilon(K_m)]$ and $(x_0, \dot{x}_0) \in K_{m-1}$. Finally, for $K \subset\subset \mathbb{R}^{2n}$ and $K \subseteq K_m$ set $\varepsilon_K := \varepsilon(K_{m+1})$. □

We will call a net as in Proposition 4.3 an *asymptotic solution* of the system of differential equations (9).

Next, we establish uniform boundedness of the asymptotic solution $(x_\varepsilon)_\varepsilon$ on compact subsets of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ (as opposed to uniform boundedness solely of $t \to x_\varepsilon(x_0, \dot{x}_0, t)$ on compact subsets of $\mathbb{R}$, as yielded by Lemma 4.2), a crucial ingredient for our proof of moderateness of the generalized coordinate transformation in Section 5.

4.4. Proposition. *The asymptotic solution* $(x_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)^{(0,1]}$ *is uniformly bounded on compact subsets of* $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$.

Proof. Let $K \times L \times J \subset\subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $\varepsilon \leq \varepsilon_{K\times L}$. Then $\varepsilon \leq \frac{1}{2}\alpha(x_0, \dot{x}_0)$ for all $(x_0, \dot{x}_0) \in K \times L$, hence $[-1, \varepsilon] \subseteq [-1, \alpha(x_0, \dot{x}_0) - \varepsilon]$. Consequently, on $K \times L \times \mathbb{R}$ the function x_ε can be written as

$$x_\varepsilon(x_0, \dot{x}_0, t) =$$
$$\begin{cases} x_0 + \dot{x}_0(t+1) + \int_{-1}^t \int_{-1}^s h(r)dr\, ds, & t \in (-\infty, -1] \\ x_0 + \dot{x}_0(t+1) + \int_{-\varepsilon}^t \int_{-\varepsilon}^s g(x_\varepsilon(x_0, \dot{x}_0, r))\delta_\varepsilon(r)dr\, ds & \\ \qquad + \int_{-1}^t \int_{-1}^s h(r)dr\, ds, & t \in [-1, \varepsilon] \\ x_\varepsilon(x_0, \dot{x}_0, \varepsilon) + \dot{x}_\varepsilon(x_0, \dot{x}_0, \varepsilon)(t - \varepsilon) + \int_\varepsilon^t \int_\varepsilon^s h(r)dr\, ds, & t \in [\varepsilon, \infty) \end{cases}.$$

Now the proof of the uniform estimates proceeds analogously to the (straightforward) boundedness proof of [13, Lemma 5.3.1], with $\sup_{x_0 \in K} |x_0|$, $\sup_{\dot{x}_0 \in K} |\dot{x}_0|$, $I(K \times L)$ playing the respective former roles of $|x_0|$, $|\dot{x}_0|$, I. □

The following result of (Kunzinger and) Steinbauer establishes existence and uniqueness of generalized geodesics ([13, Theorem 5.3.2]; compare [25, 16, 15]).

4.5. Theorem. *Let $[(\delta_\varepsilon)_\varepsilon]$ be a strict delta function, $f \in \mathrm{C}^\infty(\mathbb{R}^2, \mathbb{R})$ and let $x_0^1, \dot{x}_0^1, x_0^2, \dot{x}_0^2, v_0, \dot{v}_0 \in \mathbb{R}$. Then the system of generalized differential equations given (at the level of representatives) by*

$$\begin{aligned} \ddot{x}_\varepsilon^i(u) &= \frac{1}{2}\partial_i f(x_\varepsilon^1(u), x_\varepsilon^2(u))\, \delta_\varepsilon(u) \\ \ddot{v}_\varepsilon(u) &= f(x_\varepsilon^1(u), x_\varepsilon^2(u))\, \dot{\delta}_\varepsilon(u) + 2\sum_{i=1}^2 \partial_i f(x_\varepsilon^1(u), x_\varepsilon^2(u))\, \dot{x}_\varepsilon^i(u)\, \delta_\varepsilon(u) \end{aligned} \tag{10}$$

with initial conditions

$$x_\varepsilon^i(-1) = x_0^i, \quad \dot{x}_\varepsilon^i(-1) = \dot{x}_0^i, \quad v_\varepsilon(-1) = v_0, \quad \dot{v}_\varepsilon(-1) = \dot{v}_0$$

has a unique, c-bounded solution $\big([(x_\varepsilon^1)_\varepsilon], [(x_\varepsilon^2)_\varepsilon], [(v_\varepsilon)_\varepsilon]\big) \in \mathcal{G}(\mathbb{R})^3$. Hence, $\gamma : u \mapsto ([(x_\varepsilon^1)_\varepsilon], [(x_\varepsilon^2)_\varepsilon], [(v_\varepsilon)_\varepsilon], u)(u) \in \mathcal{G}[\mathbb{R}, \mathbb{R}^4]$ is the unique solution to the geodesic equation for the generalized metric (7)*. Furthermore, $(x_\varepsilon^i, v_\varepsilon)$ can be chosen such as to solve* (10) *classically for ε sufficiently small.*

The asymptotic solution constructed in Proposition 4.3 is a representative of the generalized solution of (10). Observe that the latter actually deserves the name "solution", despite all the subtleties of the glueing process employed in Proposition 4.3: Due to the form of the ideal $\mathcal{N}$, it is sufficient for equations to hold in $\mathcal{G}$ if they are satisfied "only" for small ε on compact sets on the level of representatives.

According to [13, 5.3.3] resp. [16, Theorem 3], the distributional limit of the solution of (10), i.e., the distribution associated to $\big([(x_\varepsilon^1)_\varepsilon], [(x_\varepsilon^2)_\varepsilon], [(v_\varepsilon)_\varepsilon]\big)$ in Theorem 4.5 is given by

$$\begin{aligned} x_\varepsilon^i(u) &\approx x_0^i + \dot{x}_0^i\,(1+u) + \frac{1}{2}\,\partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\, u_+ \\ v_\varepsilon(u) &\approx v_0 + \dot{v}_0\,(1+u) + f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\, H(u) \\ &\qquad + \sum_{i=1}^2 \partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\Big(\dot{x}_0^i + \frac{1}{4}\,\partial_i f(x_0^1 + \dot{x}_0^1, x_0^2 + \dot{x}_0^2)\Big)\, u_+. \end{aligned} \tag{11}$$

This reproduces in a rigorous way the "solutions" of (5) obtained by *ad hoc* multiplication rules in [11, 3].

5 The generalized coordinate transformation

In this section, we will define the generalized coordinate transformation $T = [(t_\varepsilon)_\varepsilon]$ modelling (3) and establish its c-boundedness as an element of $\mathcal{G}(\mathbb{R}^4)^4$. Following [15, p. 1256] resp. [13, p. 464], t_ε is obtained by taking certain geodesics of the regularized version of (1) (i.e., solutions of (10)) as new coordinate lines. More precisely, we have to pick those geodesics having vanishing initial speed in the x^1, x^2 and v-directions. Therefore we set

$$x^i_\varepsilon(-1) = x^i_0, \quad \dot{x}^i_\varepsilon(-1) = 0, \quad v_\varepsilon(-1) = v_0, \quad \dot{v}_\varepsilon(-1) = 0. \tag{12}$$

Let $(x^i_\varepsilon)_\varepsilon$ be the asymptotic solution of the first line of (10) with initial conditions (12) obtained by Proposition 4.3. Using x^i_ε in the second line of (10) yields an asymptotic solution for the entire system of differential equations. Thus, we may define the net of transformations $(t_\varepsilon)_\varepsilon$ by $t_\varepsilon := (u, x^1_\varepsilon, x^2_\varepsilon, v_\varepsilon) : \mathbb{R}^4 \to \mathbb{R}^4$,

$$t_\varepsilon : \begin{pmatrix} U \\ X^k \\ V \end{pmatrix} \mapsto \begin{pmatrix} U \\ x^i_\varepsilon(X^k, U) \\ v_\varepsilon(X^k, V, U) \end{pmatrix},$$

where $(X^k) = (X^1, X^2)$ and x^i_ε and v_ε are given implicitly (with (X^1, X^2) in a compact subset of $\mathbb{R}^2$ and for sufficiently small ε) by

$$x^i_\varepsilon(X^k, U) = X^i + \frac{1}{2}\int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \partial_i f(x^j_\varepsilon(X^k, r))\, \delta_\varepsilon(r)\, dr\, ds, \tag{13}$$

$$\begin{aligned} v_\varepsilon(X^k, V, U) = V \; &+ \int_{-\varepsilon}^{U} f(x^j_\varepsilon(X^k, s))\, \delta_\varepsilon(s)\, ds \\ &+ \int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \sum_{i=1}^{2} \partial_i f(x^j_\varepsilon(X^k, r))\, \dot{x}^i_\varepsilon(X^k, r)\, \delta_\varepsilon(r)\, dr\, ds. \end{aligned} \tag{14}$$

The “discontinuous coordinate transformation” (3) will from now on be denoted by $t := (u, x^1, x^2, v) : \mathbb{R}^4 \to \mathbb{R}^4$. Recall that it is given by

$$t : \begin{pmatrix} U \\ X^k \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) & = U \\ x^i(X^k, U) & = X^i + \frac{1}{2}\partial_i f(X^k)\, U_+ \\ v(X^k, V, U) & = V + f(X^k)\, H(U) + \frac{1}{4}\sum_{i=1}^{2} \partial_i f(X^k)^2\, U_+ \end{pmatrix}.$$

The following proposition provides the necessary $\mathcal{E}_M$-estimates and uniform bounds for t_ε resp. its components showing, in particular, that $(t_\varepsilon)_\varepsilon$ resp. T are c-bounded from $\mathbb{R}^4$ into $\mathbb{R}^4$. Relevant techniques of proof have essentially been developed by Kunzinger and Steinbauer: Starting from the uniform bounds for x_ε and $\dot{x}_\varepsilon$ provided by Proposition 4.4, derivatives with respect to U are handled by induction using the geodesic equations, while for derivatives with respect to X^i an argument involving Gronwall's Lemma is employed (see part (iv) of the proof of the following proposition). Actually, the latter method was used by Kunzinger and Steinbauer in a different context, cf. [15, p. 1258] resp. [13, Theorem 5.3.6]; we will have to deal with that issue below in Proposition 6.4.

Note that Theorem 4.5 (due to Kunzinger and Steinbauer) establishes (moderateness and) c-boundedness of the solutions of the geodesic equations (hence,

of the component functions of t_ε) only for fixed initual values, i.e., by viewing the solutions as functions depending solely on the real variable u. Consequently, only derivatives with respect to U and only compact sets of the form $K \times \{(x_0, \dot{x}_0, v_0)\}$ (with $K \subset\subset \mathbb{R}$) are taken into account in the c-boundedness estimates. As mentioned already in Section 4 when discussing the application of Lemma 4.2 to the geodesic equations, this point of view is not sufficient for our present purpose: We definitely need to consider (the component functions of) t_ε as depending on four real variables simultaneously.

In what follows, we will use the following abbreviations for partial differentiation operators: $\partial_U := \frac{\partial}{\partial U}$; $\partial_{X^j} := \frac{\partial}{\partial X^j}$; $\partial_X^\alpha := \partial_{X^1}^{\alpha_1}\partial_{X^2}^{\alpha_2}$ where $j \in \{1,2\}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$. For a detailed proof of the following proposition we refer to [9, Proposition 4.7].

5.1. Proposition. *$T = [(t_\varepsilon)_\varepsilon]$ is an element of $\mathcal{G}[\mathbb{R}^4, \mathbb{R}^4]$. Furthermore, $(\partial_X^\alpha x_\varepsilon^i)_\varepsilon$ and $(\partial_X^\alpha \partial_U x_\varepsilon^i)_\varepsilon$ are c-bounded from $\mathbb{R}^3$ into $\mathbb{R}$ for $\alpha \in \mathbb{N}_0^2$ and $i = 1, 2$.*

Proof. The proof is split into parts (i)-(viii) which altogether establish all claims. All estimates are to be understood to hold true for small ε.

(i) By Proposition 4.4, x_ε^i is uniformly bounded on compact subsets of $\mathbb{R}^4$.

(ii) Differentiating (13) with respect to U leads to a uniform estimate on compact subsets of $\mathbb{R}^4$ also for $\partial_U x_\varepsilon^i$.

(iii) Now the geodesic equation for x_ε^i inductively yields $\mathcal{E}_M$-estimates for $\partial_U^k x_\varepsilon^i$ for $k \geq 2$.

(iv) In order to estimate

$$\partial_{X^j} x_\varepsilon^i(X^k, U) = \delta_j^i + \frac{1}{2}\int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \sum_{m=1}^{2} \partial_m\partial_i f(x_\varepsilon^l(X^k, r))\, \partial_{X^j} x_\varepsilon^m(X^k, r)\, \delta_\varepsilon(r)\, dr ds \tag{15}$$

on some compact subset $K \times [-1, u_0]$ of $\mathbb{R}^3$ we define (following [15])

$$g_\varepsilon(K, u_0) := \sup\big\{\sum_{i=1}^{2} \big|\partial_{X^j} x_\varepsilon^i(X^k, U)\big| \,|\, (X^k, U) \in K \times [-1, u_0],\ j = 1, 2\big\}.$$

From (15) we obtain an estimate of the form

$$|g_\varepsilon(K, u_0)| \leq 1 + C\, C_{K,u_0} \int_{-\varepsilon}^{u_0} |g_\varepsilon(K, s)|\, ds,$$

where C_{K,u_0} is the supremum of $|\partial_i\partial_j f(x_\varepsilon^l(X^k, U))|$ with (X^k, U) ranging over $K \times [-1, u_0]$, $i, j \in \{1, 2\}$ and C is the constant from 4.1 (3). Gronwall's Lemma now implies that for small ε, $\partial_{X^j} x_\varepsilon^i$ remains uniformly bounded on compact subsets of $\mathbb{R}^3$ (note that $\partial_{X^j} x_\varepsilon^i(X^k, U) = \delta_j^i$ for $U \leq -\varepsilon$).

(v) By induction, we obtain uniform estimates for higher order derivatives with respect to X: For $\alpha \in \mathbb{N}_0^2$ with $|\alpha| \geq 2$, a somewhat involved calculation gives

$$|\partial_{X^j}\partial_X^\alpha x_\varepsilon^i(X^k, U)| \leq C_1 + \frac{1}{2}C_2 \int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} |\delta_\varepsilon(r)| \sum_{m=1}^{2} |\partial_{X^j}\partial_X^\alpha x_\varepsilon^m(X^k, r)|\, dr\, ds$$

where (X^k, U) ranges over some compact set, C_1, C_2 are positive constants and ε is sufficiently small. Estimating in a way similar to the case $|\alpha| = 1$ yields that also $\partial_{X^j}\partial_X^\alpha x_\varepsilon^i$ is uniformly bounded on compact subsets of $\mathbb{R}^3$.

(vi) From

$$\partial_X^\alpha \partial_U x_\varepsilon^i(X^k, U) = \frac{1}{2}\int_{-\varepsilon}^{U} \partial_X^\alpha \big(\partial_i f(x_\varepsilon^j(X^k, s))\big)\delta_\varepsilon(s)\, ds \tag{16}$$

one obtains uniform bounds (on compact sets) for $\partial_X^\alpha \partial_U x_\varepsilon^i$.

(vii) $\mathcal{E}_M$-estimates for $\partial_X^\alpha \partial_U^m x_\varepsilon^i$ (where $m \geq 2$) follow inductively by differentiating (16) with respect to U.

(viii) C-boundedness of $(v_\varepsilon)_\varepsilon$ is a direct consequence of the c-boundedness of $(x_\varepsilon^i)_\varepsilon$ and $(\partial_U x_\varepsilon^i)_\varepsilon$, taking into account condition 4.1 (3) on δ_ε.

Altogether, (i)–(vii) result in $(x_\varepsilon^i)_\varepsilon$ being c-bounded from $\mathbb{R}^4$ to $\mathbb{R}$ (including V as a dummy variable); (iv)–(vii) establish $(\partial_X^\alpha x_\varepsilon^i)_\varepsilon$ as c-bounded, as (vi)–(vii) do for $(\partial_X^\alpha \partial_U x_\varepsilon^i)_\varepsilon$. C-boundedness of $(v_\varepsilon)_\varepsilon$, finally, is accomplished by (viii). □

The last part of [13, Theorem 5.3.6] seems to partly anticipate Proposition 5.1 by stating that $T = [(t_\varepsilon)_\varepsilon]$ is c-bounded from some open subset of $\mathbb{R}^4$ into $\mathbb{R}^4$. In the respective proof, however, this claim is covered solely by the remark "[...] is immediate from Lemma 5.3.1"(Lemma 4.2 in this article). In view of the length of the (already fairly compact) proof of Proposition 5.1, the remark from the proof of [13, Theorem 5.3.6] cited above suggests an oversight on the part of the authors. Be that as it may, we decided to include a condensed version of the proof of Proposition 5.1 for the sake of completeness.

6 Injectivity

In this section we will show injectivity of the "discontinuous coordinate transformation" t and the functions t_ε of the generalized transformation, each on suitable subsets of $\mathbb{R}^4$. Moreover, the Jacobian determinant of t_ε will be proved to be strictly non-zero on these sets.

For classical functions, injectivity obviously is a necessary condition for being invertible. In the appropriate sense, this also holds true for (Colombeau) generalized functions ([10, Proposition 4.5]). Hence it is natural to have the present section in this article. Yet there is another reason, much deeper than the previous one, why injectivity is needed to establish T as invertible: In order to prove "asymptotic stability" of image sets under $(t_\varepsilon)_\varepsilon$, i.e., to show that there exist open sets P such that the family $(t_\varepsilon(P))_\varepsilon$ intersects with non-empty interior, we are going to employ a stability theorem due to the first author ([10, Theorem 4.6]) which, in turn, is based on a theorem of Brouwer ([18, Theorem 7.12]). Brouwer's theorem has injectivity of the functions involved among its assumptions.

In order to turn four-vectors into three-vectors we introduce the following notation: For any $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ $(n \geq 2)$, set $\hat{x} := (x^1, \dots, x^{n-1})$, and for functions f from some set into $\mathbb{R}^n$, $f = (f^1, \dots, f^n)$, set $\hat{f} := (f^1, \dots, f^{n-1})$. If $f = (f^1, \dots, f^n)$ is a function of $x = (x^1, \dots, x^n)$ with only f^n actually depending on x^n, we will not formally distinguish between $\hat{f}$ considered as a function of x (n variables) and of $\hat{x}$ ($n-1$ variables). The respective meaning will be clear from the context.

For $\hat{t}$ (hence for t), injectivity on some open set containing the half space $(-\infty, 0] \times \mathbb{R}^2$ is established by the following lemma, setting $g = \frac{1}{2}\mathrm{D}f$. Two examples will then show that in the special case $f(X, Y) = X^2 - Y^2$ considered

by Penrose in [21] such a neighbourhood is given by $(-\infty, 1) \times \mathbb{R}^2$, whereas for general (smooth) f a rectangular set of injectivity, i.e. one of the form $(-\alpha, \beta) \times \mathbb{R}^2$ with $\alpha, \beta > 0$, does not necessarily exist.

6.1. Lemma. *Let*

$$\begin{aligned} F: \quad (-a,b) \times \mathbb{R}^n \quad &\to \quad (-a,b) \times \mathbb{R}^n \\ \begin{pmatrix} U \\ X \end{pmatrix} \quad &\mapsto \quad \begin{pmatrix} U \\ X + g(X)\, U_+ \end{pmatrix}. \end{aligned}$$

where $a, b \in \mathbb{R}^+ \cup \{\infty\}$ *and* $g \in \mathrm{C}^1(\mathbb{R}^n, \mathbb{R}^n)$*. Then there exists an open set* W *containing* $(-a, 0] \times \mathbb{R}^n$ *such that* $F|_W$ *is injective.*

Proof. For $X \in \mathbb{R}^n$ define $h(X) := \sup_{z \in \overline{\mathrm{B}_{|X|}(0)}} \|\mathrm{D}g(z)\|$. The function h is continuous, non-negative and non-decreasing with $|X|$. Now set

$$W := \left\{ (U, X) \in (-a, b) \times \mathbb{R}^n \;\middle|\; -a < U < \min\Big(b, \frac{1}{h(X)}\Big) \right\}$$

(here we use the convention $\frac{1}{0} := \infty$). Let (U_1, X_1), $(U_2, X_2) \in W$ and $F(U_1, X_1) = F(U_2, X_2)$. Then $U_1 = U_2 =: U$ and $U < \frac{1}{h(X_i)}$ for $i = 1, 2$. For $U \leq 0$, we immediately obtain $X_1 = X_2$. Now let $U > 0$ and assume $X_1 \neq X_2$ with $|X_1| \geq |X_2|$, w.l.o.g. Then

$$|X_1 - X_2| = U \cdot |g(X_2) - g(X_1)| \leq U \cdot \sup_{z \in \overline{\mathrm{B}_{|X_1|}(0)}} \|\mathrm{D}g(z)\| \cdot |X_2 - X_1| < |X_2 - X_1|,$$

thereby concluding the proof by contradiction. □

In the following two examples, we consider F as in Lemma 6.1, with g being given as $\frac{1}{2}\mathrm{D}f$ for certain functions $f : \mathbb{R}^2 \to \mathbb{R}$. The map F then represents $\hat{t}$, i.e. the first three components of t corresponding to the function f at hand.

6.2. Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(X, Y) := X^2 - Y^2$. This special case was considered by Penrose in [21] (cp. also [13], components 1,2,4 of (5.45) on p. 463). In this case, an easy computation shows that $\hat{t}$ is injective (even) on $(-\infty, 1) \times \mathbb{R}^2$. The value 1 is maximal since $\hat{t}(1, X, Y_1) = (1, 2X, 0) = \hat{t}(1, X, Y_2)$ for all $X, Y_1, Y_2 \in \mathbb{R}$.

6.3. Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(X, Y) := -\frac{1}{2}(X^4 + Y^4)$. For every $\eta > 0$ the function $\hat{t}$ is non-injective on $\{\eta\} \times \mathbb{R}^2$ since $(\eta, 0, 0) = \hat{t}(\eta, 0, 0) = \hat{t}(\eta, \frac{1}{\sqrt{\eta}}, \frac{1}{\sqrt{\eta}}) = \hat{t}(\eta, -\frac{1}{\sqrt{\eta}}, -\frac{1}{\sqrt{\eta}})$. Hence, on every set of the form $(-\alpha, \beta) \times \mathbb{R}^2$ $(\alpha, \beta > 0)$, $\hat{t}$ is non-injective. However, $\hat{t}$ is injective on $W = \{(U, X, Y) \,|\, U < \frac{1}{3}(X^2 + Y^2)^{-1}\}$.

We now turn to the question of injectivity of the generalized coordinate transformation. In [15] (cf. also [13, Theorem 5.3.6]) Kunzinger and Steinbauer claim that for sufficiently small ε, the functions t_ε are diffeomorphisms on a suitable *rectangular* open subset Ω of $\mathbb{R}^4$ containing the shock hyperplane $U = 0$. In their proof, they employ a global univalence theorem of Gale and Nikaido, stating that any differentiable function $F : \Omega \to \mathbb{R}^n$, where Ω is a closed rectangular region in $\mathbb{R}^n$, is injective if all principal minors of its Jacobian $J(x)$

are positive (see [12]). However, a closer look at the proofs in [15] resp. of [13, Theorem 5.3.6] reveals that the condition of Gale and Nikaido's Theorem in fact is established only on sets of the form $(-\infty,\eta\,]\times K\times\mathbb{R}$ for sufficiently small ε, say $\varepsilon\le\varepsilon_0$, where K is a compact subset of $\mathbb{R}^2$ and η and ε_0 both depend on K. Furthermore, they use uniform boundedness of $(x^i_\varepsilon)_\varepsilon$ on compact subsets of $\mathbb{R}^4$ (Proposition 4.4) whereas [13, Lemma 5.3.1] (Lemma 4.2 above) only provides boundedness on compact subsets of $\mathbb{R}$ for fixed initial values x^i_0 and $\dot x^i_0$.

Therefore, we restate Theorem 5.3.6 of [13], keeping only those claims which are actually shown in [15] resp. [13] and complementing it with a sketch of proof.

6.4. Proposition. *For every $K\subset\subset\mathbb{R}^2$ and $\delta>0$ there exist $\eta>0$ and $\varepsilon_0\in(0,1]$ such that every principal minor of $\mathrm{D}t_\varepsilon(U,X^i,V)$ stays in $(1-\delta,1+\delta)$ for all $(U,X^i,V)\in(-\infty,\eta\,]\times K\times\mathbb{R}$ and $\varepsilon\le\varepsilon_0$. In particular, $\det\circ\mathrm{D}T$ is strictly non-zero on $(-\infty,\eta\,]\times K\times\mathbb{R}$ and every principal minor of $\mathrm{D}t_\varepsilon(U,X^i,V)$ is positive for $(U,X^i,V)\in(-\infty,\eta\,]\times K\times\mathbb{R}$ and $\varepsilon\le\varepsilon_0$.*

Proof. We have to find estimates for $\frac{\partial x^i_\varepsilon}{\partial X^j}(X^k,U)-\delta^i_j$; the partial derivative can be written as in (15). Noting that $\frac{\partial x^i_\varepsilon}{\partial X^j}(X^k,U)=\delta^i_j$ for $U\le-\varepsilon$, we obtain

$$\left|\frac{\partial x^i_\varepsilon}{\partial X^j}(X^k,U)-\delta^i_j\right|\le C\,C_{K,1}\,C_1\,(U+\varepsilon)_+ \tag{17}$$

for $(X^k,U)\in K\times(-\infty,1]$ and sufficiently small ε. Here, C_1 is a constant chosen according to the c-boundedness of $\big(\frac{\partial x^i_\varepsilon}{\partial X^j}\big)_\varepsilon$ and $C_{K,1}$ has the same meaning as in the proof of Proposition 5.1. Thus the supremum of the left hand side of (17) for $(X^k,U)\in K\times(-\infty,\eta]$ stays arbitrarily close to 0 for all $\varepsilon\le\varepsilon_0$ if $\eta>0$ and $\varepsilon_0\in(0,1]$ are chosen accordingly. ☐

We will say a smooth net $(u_\varepsilon)_\varepsilon:(-a,b)\times\mathbb{R}^n\times\mathbb{R}\to(-a,b)\times\mathbb{R}^n\times\mathbb{R}$ (for $a,b\in\mathbb{R}^+\cup\{\infty\}$) has *property (E)* if for every compact subset K of $\mathbb{R}^n$ there exist $\alpha\in(0,b)$ and $\varepsilon_0\in(0,1]$ such that u_ε is injective on $(-a,\alpha]\times K\times\mathbb{R}$ for all $\varepsilon\le\varepsilon_0$. The net $(u_\varepsilon)_\varepsilon$ is said to have *property (E+)* if for every compact subset K of $\mathbb{R}^n$ there exist $\alpha\in(0,b)$ and $\varepsilon_0\in(0,1]$ such that u_ε is injective on $(-a,\alpha]\times K\times\mathbb{R}$ and $(\det\circ\mathrm{D}u_\varepsilon)_\varepsilon$ is strictly non-zero, uniformly on $(-a,\alpha]\times K\times\mathbb{R}$ for all $\varepsilon\le\varepsilon_0$, i.e. an estimate as (6) holds for all $(U,X,V)\in(-a,\alpha]\times K\times\mathbb{R}$.

Combining the preceding proposition and the univalence theorem of Gale and Nikaido, it follows that $(t_\varepsilon)_\varepsilon$ has property (E+). By [9, Theorem 3.59], this is already sufficient for T to be left invertible in the sense of Definition 3.4:

6.5. Corollary. *For every open relatively compact subset W of $\mathbb{R}^2$ there exists some $\alpha>0$ such that for all $\beta>0$ and for all bounded open intervals I the generalized function T is left invertible on $(-\beta,\alpha)\times W\times I$.*

7 Uniform convergence

The key idea for showing that the images of certain sets under the t_ε intersect with non-empty interior consists in observing that if t_ε stays close enough to t, then also the image of some set W under t_ε stays close to $t(W)$. Therefore, convergence of $(t_\varepsilon)_\varepsilon$ to t as $\varepsilon\to0$ in some sense might be useful. The last statement of [13, Theorem 5.3.3] (cf. also [16, Theorem 3]) tells us that

$x^i_\varepsilon(\,.\,,X^1,X^2,V)$ converges to $x^i(\,.\,,X^1,X^2,V)$ as $\varepsilon \to 0$, yet only in the sense of uniform convergence on compact subsets of $\mathbb{R}$ for fixed $(X^1,X^2,V) \in \mathbb{R}^3$. In contrast, we will need (and establish in the sequel) uniform convergence of $(x^i_\varepsilon)_\varepsilon$ to x^i on arbitrary compact subsets of $\mathbb{R}^4$. Obviously, this is impossible for v_ε since v is discontinuous. However, cutting out the part of v_ε converging (pointwise for $U \neq 0$) to the term involving the Heaviside function, we again can prove uniform convergence on arbitrary compact sets. To this end, we define

$$\begin{aligned} w(X^k,V,U) &:= V + \frac{1}{4}\sum_{i=1}^{2} \partial_i f(X^k)^2\, U_+,\\ w_\varepsilon(X^k,V,U) &:= V + \int_{-\varepsilon}^{U}\int_{-\varepsilon}^{s} \sum_{i=1}^{2} \partial_i f(x^j_\varepsilon(X^k,r))\, \dot{x}^i_\varepsilon(X^k,r)\, \delta_\varepsilon(r)\, dr\, ds. \end{aligned}$$

Furthermore, let $s := (u,x^1,x^2,w)$ and $s_\varepsilon := (u,x^1_\varepsilon,x^2_\varepsilon,w_\varepsilon)$. Obviously, $\hat{t} = \hat{s}$, implying that also $\hat{s}$ is injective on some open set containing the half space $(-\infty,0] \times \mathbb{R}^2$. Moreover, since all principal minors of $\mathrm{D}t_\varepsilon$ are independent of the derivatives of v_ε, Proposition 6.4 also holds for $(s_\varepsilon)_\varepsilon$. Therefore, also $(s_\varepsilon)_\varepsilon$ has property (E+).

In a first step, we state that $\dot{t}_\varepsilon \to \dot{t}$ and, due to the same proof, $\dot{s}_\varepsilon \to \dot{s}$, uniformly on compact subsets of $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}^3$ for $\varepsilon \to 0$. The proof proceeds along the same lines as the proof of [13, Theorem 5.3.3]. For the detailed argument we refer to [9, Lemma 4.13].

The formal similarity of the respective proofs of [13, Theorem 5.3.3] and Proposition 7.1 is owed to the fact that the former determines the limits of the right hand sides of (10), integrated against a test function ψ, while the latter establishes the limits (uniformly on compact sets) of the derivatives of the right hand sides of (13) and (14). Now, in fact, (10) is but the second derivative of (13)(14).

7.1. Proposition. *$\dot{t}_\varepsilon \to \dot{t}$ as $\varepsilon \to 0$, uniformly on compact subsets of $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}^3$.*

In order to pass from $\dot{s}_\varepsilon \to \dot{s}$ to $s_\varepsilon \to s$, we employ the following auxiliary result [9, Lemma 4.14].

7.2. Lemma. *Let $f_\varepsilon, f \in \mathrm{C}(\mathbb{R}^n,\mathbb{R})$ (for $\varepsilon \in (0,1]$). Suppose that $\partial_n f_\varepsilon(x,t)$ and $\partial_n f(x,t)$ exist for all $(x,t) \in \mathbb{R}^{n-1} \times (\mathbb{R}\backslash\{0\})$ and that $\partial_n f_\varepsilon(x,\,.\,)$ and $\partial_n f(x,\,.\,)$ are piecewise continuous (with one-sided limits existing) for all $x \in \mathbb{R}^{n-1}$. Let $c \in \mathbb{R}$ with $c < 0$. If*

(1) $f_\varepsilon \to f$ for $\varepsilon \to 0$ uniformly on $K \times \{c\}$ for all $K \subset\subset \mathbb{R}^{n-1}$,

(2) $\partial_n f_\varepsilon \to \partial_n f$ for $\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{R}^{n-1} \times (\mathbb{R}\backslash\{0\})$, and

(3) $\|\partial_n f_\varepsilon - \partial_n f\|_{\infty,K\times([-d,d]\backslash\{0\})}$ is uniformly bounded for any compact set $K \subset\subset \mathbb{R}^{n-1}$ and some $d > 0$,

then $f_\varepsilon \to f$ for $\varepsilon \to 0$ uniformly on arbitrary compact subsets of $\mathbb{R}^n$.

Now we are ready to prove

7.3. Proposition. *$s_\varepsilon \to s$ for $\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{R}^4$.*

Proof. We show that for each component function of s_ε the conditions of Lemma 7.2 are satisfied with respect to s. The symbol ∂_n in Lemma 7.2, if applied to x^i_ε resp. w_ε, is understood to denote the derivatives of x^i_ε resp. w_ε with respect to U.

$\dot{x}^i_\varepsilon$ resp. $\dot{w}_\varepsilon$ are smooth on $\mathbb{R}^3$ resp. $\mathbb{R}^4$, x^i and w are smooth on $\mathbb{R}^2 \times (\mathbb{R}\backslash\{0\})$ resp. $\mathbb{R}^3 \times (\mathbb{R}\backslash\{0\})$. $\dot{x}^i(X^1, X^2, \,.\,)$ and $\dot{w}(X^1, X^2, V, \,.\,)$ are piecewise continuous for all $(X^1, X^2) \in \mathbb{R}^2$ resp. $(X^1, X^2, V) \in \mathbb{R}^3$. For $U = -1$ the integral terms of $x^i_\varepsilon(\,.\,,\,.\,, U)$ and $w_\varepsilon(\,.\,,\,.\,,\,.\,, U)$ vanish and $x^i_\varepsilon = x^i$ and $w_\varepsilon = w$. Hence, condition (1) is satisfied. By Proposition 7.1, $\dot{x}^i_\varepsilon \to \dot{x}^i$ and $\dot{w}_\varepsilon \to \dot{w}$ for $\varepsilon \to 0$ uniformly on compact subsets of $\mathbb{R}^2 \times (\mathbb{R}\backslash\{0\})$ resp. $\mathbb{R}^3 \times (\mathbb{R}\backslash\{0\})$, i.e. they satisfy condition (2). Finally, by Theorem 4.5, $\dot{x}^i_\varepsilon$ is uniformly bounded on compact sets and, therefore, this is also true for $\dot{w}_\varepsilon$. Since both $\dot{x}^i$ and $\dot{w}$ are bounded on any bounded subset of $\mathbb{R}^2 \times (\mathbb{R}\backslash\{0\})$ resp. $\mathbb{R}^3 \times (\mathbb{R}\backslash\{0\})$, also condition (3) is satisfied and the claim follows. □

8 Inversion of the generalized coordinate transformation

Finally, we turn to establishing local invertibility of the generalized coordinate transformation T. The core of the proof consists in showing that there exist open sets P such that, for ε small, the intersection of the $t_\varepsilon(P)$ has non-empty interior. The sets P can be chosen such as to contain arbitrarily large (bounded) portions of the left half space $U \leq 0$.

The achievements of Kunzinger and Steinbauer in the context of inverting T have already been discussed in some detail in previous sections; recall, in particular, what has been said in Sections 1, 4 and 6.

In the sequel, we will often have to make use of cylinders rather than balls. Therefore, for $x = (\hat{x}, x^n) \in \mathbb{R}^n$, let $\mathrm{B}^{\mathrm{Z}}_{\delta,\eta}(x)$ denote the cylinder $\mathrm{B}_\delta(\hat{x}) \times (x^n - \eta, x^n + \eta)$. Theorem 8.1 below, being one of this section's prominent technical tools, arises as a slightly modified version of [10, Theorem 4.6] where the open balls $\mathrm{B}_\delta(0)$ are replaced by cylinders $\mathrm{B}^{\mathrm{Z}}_{\delta,\eta}(0)$. We leave it to the reader to adapt the proof of [10, Theorem 4.6] to the case of cylinders. Roughly speaking, this "stability theorem" establishes a kind of continuous dependence of connected parts $f(A)$ of the image set $f(U)$ on the function f.

8.1. Theorem. *Let U be an open subset of $\mathbb{R}^n$, $f, g \in \mathrm{C}(U, \mathbb{R}^n)$ both injective and W a connected open subset of $\mathbb{R}^n$ with $\overline{W} \subset\subset f(U)$. Choose $y \in W$ and $\delta, \eta > 0$ with $y + \mathrm{B}^{\mathrm{Z}}_{\delta,\eta}(0) \subseteq W$ such that the closure of $W_{\delta,\eta} := W + \mathrm{B}^{\mathrm{Z}}_{\delta,\eta}(0)$ is still a subset of $f(U)$. If, for $A := f^{-1}(\overline{W_{\delta,\eta}})$ and $f = (\hat{f}, f^n)$ resp. $g = (\hat{g}, g^n)$, both*

$$\|\hat{g} - \hat{f}\|_{\infty,A} < \delta \quad \text{and} \quad \|g^n - f^n\|_{\infty,A} < \eta$$

hold, then

$$\overline{W} \subseteq g(A)^\circ.$$

Now we are ready to prove that the domains of suitable inverses of the t_ε intersect with non-empty interior. The following theorem yields the desired result for an entire class of c-bounded nets (also denoted by $(t_\varepsilon)_\varepsilon$) of smooth functions of which our particular $(t_\varepsilon)_\varepsilon$ at hand is but a special case.

8.2. Theorem. *Let $a, b \in \mathbb{R}^+ \cup \{\infty\}$. Let the functions t_ε, s_ε (for every $\varepsilon \in (0,1]$) and s satisfy the following assumptions:*

(1) $t_\varepsilon : (-a,b) \times \mathbb{R}^n \times \mathbb{R} \to (-a,b) \times \mathbb{R}^n \times \mathbb{R}$

$$\begin{pmatrix} U \\ X \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) & := U \\ x_\varepsilon(U,X) & \\ v_\varepsilon(U,X,V) & := V + g_\varepsilon(U,X) + h_\varepsilon(U,X) \end{pmatrix}$$

where $x_\varepsilon \in \mathrm{C}^\infty((-a,b) \times \mathbb{R}^n, \mathbb{R}^n)$ and $g_\varepsilon, h_\varepsilon \in \mathrm{C}^\infty((-a,b) \times \mathbb{R}^n, \mathbb{R})$. Assume that $(t_\varepsilon)_\varepsilon$ has property (E), i.e. that for every compact subset K of $\mathbb{R}^n$ there exist $\alpha \in (0,b)$ and $\varepsilon' \in (0,1]$ such that t_ε is injective on $(-a,\alpha] \times K \times \mathbb{R}$ for all $\varepsilon \leq \varepsilon'$. Furthermore, suppose that $(h_\varepsilon)_\varepsilon$ is uniformly bounded on compact subsets of $(-a,b) \times \mathbb{R}^n$.

(2) $s_\varepsilon : (-a,b) \times \mathbb{R}^n \times \mathbb{R} \to (-a,b) \times \mathbb{R}^n \times \mathbb{R}$

$$\begin{pmatrix} U \\ X \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) & = U \\ x_\varepsilon(U,X) & \\ w_\varepsilon(U,X,V) & := V + g_\varepsilon(U,X) \end{pmatrix}.$$

By (1), s_ε is smooth. Suppose that also $(s_\varepsilon)_\varepsilon$ has property (E).

(3) $s : (-a,b) \times \mathbb{R}^n \times \mathbb{R} \to (-a,b) \times \mathbb{R}^n \times \mathbb{R}$

$$\begin{pmatrix} U \\ X \\ V \end{pmatrix} \mapsto \begin{pmatrix} u(U) & = U \\ x(U,X) & \\ w(U,X,V) & := V + g(U,X) \end{pmatrix}$$

where $x \in \mathrm{C}((-a,b) \times \mathbb{R}^n, \mathbb{R}^n)$ and $g \in \mathrm{C}((-a,b) \times \mathbb{R}^n, \mathbb{R})$. Assume that for $\hat{s} := (u,x) : (-a,b) \times \mathbb{R}^n \to (-a,b) \times \mathbb{R}^n$, there exists some open set W containing $(-a,0] \times \mathbb{R}^n$ such that $\hat{s}|_W$ is injective.

Finally, suppose $s_\varepsilon \to s$ for $\varepsilon \to 0$ uniformly on compact sets.

Then the following holds: For every p on the hyperplane $U = 0$ there exist open neighbourhoods P of p with $P \subseteq W \times \mathbb{R}$ and Q of $q := s(p)$ with $Q \subseteq s(W \times \mathbb{R})$, and some $\varepsilon_0 \in (0,1]$ such that

$$\overline{Q} \subseteq t_\varepsilon(P)$$

for all $\varepsilon \leq \varepsilon_0$.

Proof. By a theorem of Brouwer ([18, Theorem 7.12]), $\hat{s}(W)$ is open in $\mathbb{R}^{n+1}$ and $\hat{s}|_W : W \to \hat{s}(W)$ is a homeomorphism. Note that with $\hat{s}|_W$, also $s|_{W\times\mathbb{R}}$ is a homeomorphism and that $s(W \times \mathbb{R})$ equals the open set $\hat{s}(W) \times \mathbb{R}$. We will simply write $\hat{s}$ and s in place of $\hat{s}|_W$ resp. $s|_{W\times\mathbb{R}}$. Noting that $\hat{t}_\varepsilon = \hat{s}_\varepsilon = (u, x_\varepsilon)$, we have $\hat{t}_\varepsilon = \hat{s}_\varepsilon \to \hat{s}$ uniformly on compact sets as $\varepsilon \to 0$, by Proposition 7.3.

Let $p = (0, x_p, v_p)$ be a point of the hyperplane $U = 0$, $q := s(p) = (0, x_q, v_q)$, $\hat{p} = (0, x_p)$ and $\hat{q} = \hat{s}(\hat{p}) = (0, x_q)$. Let $R \subseteq \mathbb{R}^n$ be a bounded open set satisfying $\overline{R}^\circ = R$ containing x_p. Choose $\alpha \in (0, \min(a,b))$ and $\lambda > 0$ such that $(-a,\alpha] \times \overline{R_\lambda} \subseteq W$ where $R_\lambda := R + \mathrm{B}_\lambda(0)$. Then s is injective on $(-a,\alpha] \times \overline{R_\lambda} \times \mathbb{R}$. By property (E), we can assume w.l.o.g. (making α smaller if necessary) that there exists $\varepsilon_1 \in (0,1]$ such that also $(t_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ are injective on $(-a,\alpha] \times \overline{R_\lambda} \times \mathbb{R}$

for all $\varepsilon \le \varepsilon_1$. Defining $G := (-a, \alpha) \times R_\lambda \times \mathbb{R}$, we have, in particular, that s, t_ε and s_ε (for $\varepsilon \le \varepsilon_1$) are injective on G.

Fix $\gamma \in (0, \alpha)$ and $\beta \in [\gamma, a)$. Choose $\delta > 0$ with $\hat{s}^{-1}(\overline{\mathrm{B}_{3\delta}(\hat{q})}) \subseteq (-\beta, \gamma) \times R$, i.e. $\overline{\mathrm{B}_{3\delta}(\hat{q})} \subseteq \hat{s}((-\beta, \gamma) \times R)$. Let $\mu \in (\beta, a)$. Choose $\eta \ge \delta$ and $\varepsilon_2 \le \varepsilon_1$ such that

$$\|v_\varepsilon - w_\varepsilon\|_{\infty, [-\mu, \alpha] \times \overline{R_\lambda} \times \mathbb{R}} = \|h_\varepsilon\|_{\infty, [-\mu, \alpha] \times \overline{R_\lambda}} < \eta$$

for all $\varepsilon \le \varepsilon_2$. Since $s(W \times \mathbb{R}) = \hat{s}(W) \times \mathbb{R}$, it follows that $\overline{\mathrm{B}^{\mathrm{Z}}_{3\delta, 2\eta+\delta}(q)} = \overline{\mathrm{B}_{3\delta}(\hat{q})} \times [v_q - (2\eta+\delta), v_q + (2\eta+\delta)]$ is a compact subset of $s(W \times \mathbb{R})$. Now let I be a bounded open interval in $\mathbb{R}$ such that $s^{-1}\Big(\overline{\mathrm{B}^{\mathrm{Z}}_{3\delta, 2\eta+\delta}(q)}\Big) \subseteq (-\beta, \gamma) \times R \times I =: P$ which is possible since only the last component of s is dependent on V and this dependence is a linear one. Applying s to both sides of this inclusion yields

$$\overline{\mathrm{B}^{\mathrm{Z}}_{3\delta, 2\eta+\delta}(q)} \subseteq s(P). \tag{18}$$

Observe that $p \in P \subseteq \overline{P} \subset\subset G$ and $q \in s(P)$. Again by Proposition 7.3, there exists $\varepsilon_0 \le \varepsilon_2$ such that

$$\|\hat{s}_\varepsilon - \hat{s}\|_{\infty, \overline{P}} < \frac{\delta}{2} \quad \text{and} \quad \|w_\varepsilon - w\|_{\infty, \overline{P}} < \frac{\delta}{2}$$

for all $\varepsilon \le \varepsilon_0$. The set $Q_0' := s(P) \backslash (\partial s(P) + \overline{\mathrm{B}^{\mathrm{Z}}_{\delta, \delta}(0)})$ is open and bounded since $s(P)$ has these properties. By (18) and by definition of Q_0',

$$\overline{\mathrm{B}^{\mathrm{Z}}_{2\delta, 2\eta}(q)} \subseteq Q_0' \tag{19}$$

holds. Now let Q' be the connected component of Q_0' containing q, hence also containing the (connected) set $\overline{\mathrm{B}^{\mathrm{Z}}_{2\delta, 2\eta}(q)}$. Obviously, Q' is open, bounded and connected.

Now we apply Theorem 8.1 for the first time, with G, s, s_{ε_0}, Q', q, δ, δ and $M' := s^{-1}(\overline{Q' + \mathrm{B}^{\mathrm{Z}}_{\delta, \delta}(0)})$ in place of U, f, g, W, y, δ, η and A to arrive at

$$\overline{Q'} \subseteq s_{\varepsilon_0}(M')^\circ \subseteq s_{\varepsilon_0}(\overline{P})^\circ.$$

We now set out to apply Theorem 8.1 once more to derive an analogous statement with respect to t_ε. Similarly to above, set $Q_0 := Q' \backslash (\partial Q' + \overline{\mathrm{B}^{\mathrm{Z}}_{\delta, \eta}(0)})$. Again, Q_0 is open and bounded. By (19) and with Q denoting the connected component of Q_0 containing q, we have $\overline{\mathrm{B}^{\mathrm{Z}}_{\delta, \eta}(q)} \subseteq Q$.

Applying Theorem 8.1 again, this time with respect to G, s_{ε_0}, t_ε (for fixed $\varepsilon \le \varepsilon_0$), Q, q, δ, η and $M := s_{\varepsilon_0}^{-1}(\overline{Q + \mathrm{B}^{\mathrm{Z}}_{\delta, \eta}(0)})$ in place of U, f, g, W, y, δ, η and A, we obtain

$$\overline{Q} \subseteq t_\varepsilon(M)^\circ \subseteq t_\varepsilon(\overline{P}).$$

Hence, t_ε being a homeomorphism on G and $\overline{R}^\circ = R$, the inclusion $\overline{Q} \subseteq t_\varepsilon(P)$ holds for all $\varepsilon \le \varepsilon_0$. □

8.3. Remark. An inspection of the preceding proof reveals that P can be chosen as having the form $(-\beta, \gamma) \times R \times I$ where $-\beta < 0$ is arbitrarily close to $-a$, the sets R and I are arbitrarily large, yet bounded open sets (I being of a certain minimum size depending on $\|h_\varepsilon\|_\infty$ on compact sets for small ε) and γ has to be sufficiently small, depending (via α) on R and the injectivity behaviour of s, $(t_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ for $U > 0$.

If the functions t_ε, s_ε in Theorem 8.2 are representatives of generalized functions T and S then T is invertible around any point on the shock hyperplane, provided $(t_\varepsilon)_\varepsilon$ satisfies property (E+):

8.4. Theorem. *Let $(t_\varepsilon)_\varepsilon$, $(s_\varepsilon)_\varepsilon$ and s be as in Theorem 8.2. If, in addition, $(t_\varepsilon)_\varepsilon$ has property (E+) and*

$$T := [(t_\varepsilon)_\varepsilon] \in \mathcal{G}[(-a,b)\times\mathbb{R}^n\times\mathbb{R}, (-a,b)\times\mathbb{R}^n\times\mathbb{R}]$$

and

$$S := [(s_\varepsilon)_\varepsilon] \in \mathcal{G}[(-a,b)\times\mathbb{R}^n\times\mathbb{R}, (-a,b)\times\mathbb{R}^n\times\mathbb{R}],$$

then, for every p on the hyperplane $U = 0$, there exists an open neighbourhood A of p in $(-a,b)\times\mathbb{R}^n\times\mathbb{R}$ such that T is invertible on A with inversion data $[A, \mathbb{R}^{n+2}, T^\diamond, B, Q]$ where $T^\diamond \in \mathcal{G}[\mathbb{R}^{n+2}, D]$ and B, Q and D are suitable bounded open subsets of $(-a,b)\times\mathbb{R}^n\times\mathbb{R}$ with $Q \subseteq B$ and $A \subseteq D$.

Proof. Let α, R_λ, $G = (-a,\alpha)\times R_\lambda\times\mathbb{R}$, P, Q and ε_0 be as in the proof of Theorem 8.2. Recall that under these assumptions, t_ε is injective on G and the inclusions $p \in P \subseteq \overline{P} \subset\subset G$ and $\overline{Q} \subseteq t_\varepsilon(P)$ hold. Assuming that α was chosen according to property (E+), there exist $\varepsilon' \leq \varepsilon_0$, $C' > 0$ and $N' \in \mathbb{N}$ such that

$$\inf_{(U,X,V)\in G} |\det(\mathrm{D}t_\varepsilon(U,X,V))| \geq C'\varepsilon^{N'} \tag{20}$$

for all $\varepsilon \leq \varepsilon'$. Let A and D_1 be open subsets of G such that $\overline{P} \subset\subset A \subseteq \overline{A} \subset\subset D_1 \subseteq \overline{D_1} \subset\subset G$. Then $p \in A$ and $K_\varepsilon := t_\varepsilon(\overline{A})$ is compact for all $\varepsilon \leq \varepsilon_0$. Obviously, the estimate from below in (20) is also valid for all $(U,X,V) \in D_1 \subseteq G$.

We now apply [10, Proposition 5.4] to $(-a,b)\times\mathbb{R}^n\times\mathbb{R}$, D_1, $(t_\varepsilon)_\varepsilon$, $((t_\varepsilon|_{D_1})^{-1})_\varepsilon$, p, $\{p\}$, $\overline{A}$ and K_ε (in place of U, W, $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon$, $[(\tilde{x}_\varepsilon)_\varepsilon]$, K', K and K_ε in the notation of [10]). Essentially, this (technical) proposition states the following: If a moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$ with all u_ε injective on a relatively compact open subset W of U satisfies an estimate corresponding to (20) on W, then the inverses v_ε of $u_\varepsilon|_W$ can be extended to a uniformly bounded moderate net $(\tilde{v}_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)^n$ in such a way that $\tilde{v}_\varepsilon|_{u_\varepsilon(K)} = v_\varepsilon|_{u_\varepsilon(K)}$ and $\tilde{v}_\varepsilon(x) = y_0$ on $\mathbb{R}^n\backslash u_\varepsilon(W)$, where $y_0 \in \mathbb{R}^n$ and the compact subset K of W can be arbitrarliy prescribed. Therefore, there exist extensions $t_\varepsilon^\diamond$ of $(t_\varepsilon|_{D_1})^{-1}$ with $t_\varepsilon^\diamond|_{K_\varepsilon} = ((t_\varepsilon|_{D_1})^{-1})|_{K_\varepsilon}$ and $t_\varepsilon^\diamond(x) = p$ on $\mathbb{R}^{n+2}\backslash t_\varepsilon(D_1)$ such that $(t_\varepsilon^\diamond)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^{n+2})^{n+2}$. In particular, the proposition ensures that the net $(t_\varepsilon^\diamond)_\varepsilon$ is c-bounded into any (bounded) open subset D of $\mathbb{R}^{n+2}$ that contains the convex hull of $\overline{D_1}\cup\{p\} = \overline{D_1}$. As to the last statment, see the proof of [10, Proposition 5.3 (2)] where $\tilde{v}_\varepsilon$ (i.e. $t_\varepsilon^\diamond$, in the case at hand) is explicitly constructed.

Set $T^\diamond := [(t_\varepsilon^\diamond)_\varepsilon] \in \mathcal{G}[\mathbb{R}^{n+2}, D]$. On the one hand, we have $Q \subseteq t_\varepsilon(P) \subseteq t_\varepsilon(A) \subseteq K_\varepsilon$ and, therefore, $t_\varepsilon^\diamond(Q) = (t_\varepsilon|_{D_1})^{-1}(Q) \subseteq P \subseteq \overline{P} \subset\subset A$, implying that $(t_\varepsilon^\diamond|_Q)_\varepsilon$ is c-bounded into A. Moreover,

$$t_\varepsilon \circ t_\varepsilon^\diamond|_Q = t_\varepsilon \circ t_\varepsilon^{-1}|_Q = \mathrm{id}_Q,$$

establishing $[A, \mathbb{R}^{n+2}, T^\diamond, Q]$ as a right inverse of T on A. On the other hand, since $t_\varepsilon(A) \subseteq K_\varepsilon$, we have

$$t_\varepsilon^\diamond \circ t_\varepsilon|_A = t_\varepsilon^\diamond|_{K_\varepsilon} \circ t_\varepsilon|_A = t_\varepsilon^{-1}|_{K_\varepsilon} \circ t_\varepsilon|_A = \mathrm{id}_A\,.$$

By c-boundedness of $(t_\varepsilon)_\varepsilon$, there exists $K' \subset\subset (-a, b) \times \mathbb{R}^n \times \mathbb{R}$ with $t_\varepsilon(\overline{A}) \subseteq K'$ for sufficiently small ε. Hence, $(t_\varepsilon|_A)_\varepsilon$ is c-bounded into any (bounded) open set B containing K'. It follows that $[A, \mathbb{R}^{n+2}, T^\diamond, B]$ is a left inverse of T on A. Combining these results, we obtain that T is invertible on A with inversion data $[A, \mathbb{R}^{n+2}, T^\diamond, B, Q]$. $\square$

Having collected the necessary tools, we can now establish the main result on invertibility of the generalized coordinate transformation T.

8.5. Theorem. *The generalized coordinate transformation* $T = [(t_\varepsilon)_\varepsilon]$ *is locally invertible (in the sense of Definition 3.5) on some open set* Ω *containing the half space* $(-\infty, 0] \times \mathbb{R}^3$.

Proof. By Proposition 6.4, $(t_\varepsilon)_\varepsilon$ as well as $(s_\varepsilon)_\varepsilon$ possess property (E+). Moreover, $\hat{s}$ is injective on some open set W containing $(-\infty, 0] \times \mathbb{R}^2$ by Lemma 6.1. Then, by Theorem 8.4, for every p on the hyperplane $U = 0$ there exists an open neighbourhood $A(p) \subseteq \mathbb{R}^4$ such that T is invertible on $A(p)$. Recall that each $A(p)$ contains some set $P = (-\beta, \gamma) \times R \times I$ as discussed in Remark 8.3. In particular, all of $\beta > 0$, R and I (both bounded) can be chosen arbitrarily large. Forming the union Ω of a family of $A(p)$ with the corresponding sets P covering the left half space, we obtain that the generalized function T is locally invertible on Ω, constituting an open set containing $(-\infty, 0] \times \mathbb{R}^3$. $\square$

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